Optimization

Fall semester 2022/23

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Eötvös Loránd University
Institute of Mathematics
Department of Operations Research

Set 1

Question 1

Bob would like to write down the system

$$3x + 2y + 4z = 8, \tag{1}$$

$$-3y \leqslant 3, \tag{2}$$

$$x - 3z \geqslant 10, \tag{3}$$

$$\min x - y, \tag{4}$$

but his keyboard is missing the symbols $=$ and $\geqslant$, and the letter i is not working. Reformulate the problem only using $\leqslant$ and maximization. (1pt)

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Prove that the system $Ax \leqslant 0, x \gg 0$ admits a solution if and only if $Ax \leqslant 0, x \geqslant 1$ has one. (1pt)

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Also, $Ax' = A\frac{x}{\min_i x_i} \leqslant 0$ as $Ax \leqslant 0$ and $\frac{1}{\min_i x_i} \geqslant 0$

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Consider the problem

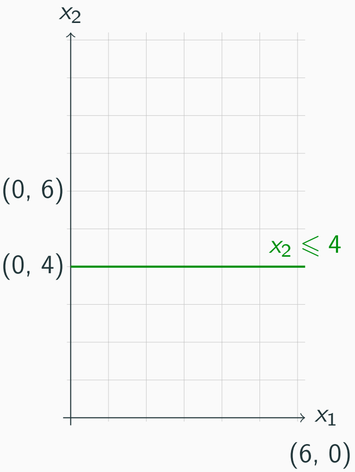
$$x_2 \leqslant 4, \quad (5)$$

$$x_1 + x_2 \leqslant 6, \quad (6)$$

$$2x_1 + x_2 \leqslant 10 \quad (7)$$

$$x_1, x_2 \geqslant 0 \quad (8)$$

Represent these constraints on the plane. Find a point that maximizes $x_1 + 2x_2$. (2pts)



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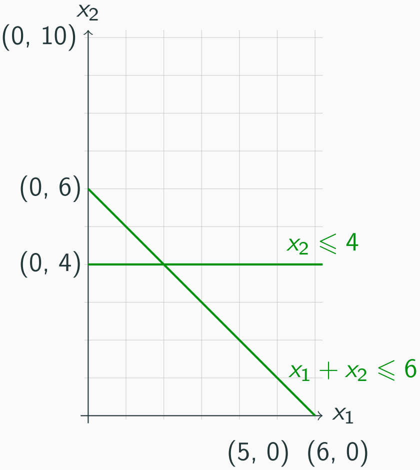

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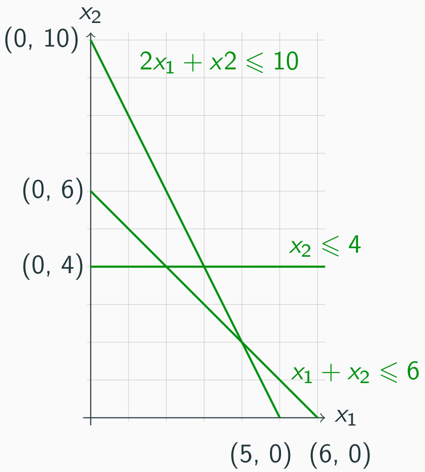

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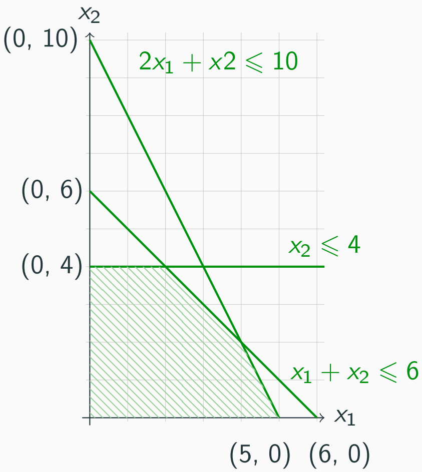

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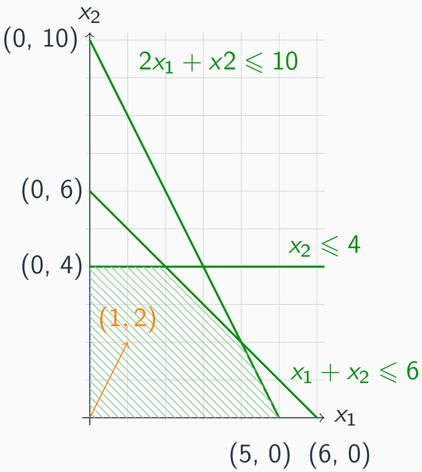

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Optimal solution: $x_1 = 2, x_2 = 4$.

Optimal value: $x_1 + 2x_2 = 2 + 2 * 4 = 10$.

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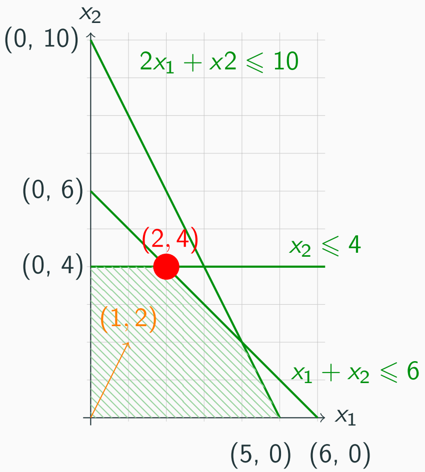

Question 3: Strong Duality Thm

Consider the problem

$$x_2 \leqslant 4, \quad (9)$$

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The dual of this problem is:

$$\begin{aligned} \min\ & 4y_1 + 6y_2 + 10y_3 \\ \text{s.t.}\quad & y_2 + 2y_3 \geqslant 1 \\ & y_1 + y_2 + y_3 \geqslant 2 \\ & y_1, y_2, y_3 \geqslant 0 \end{aligned}$$

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max $=$ min as the Strong Duality theorem states.

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Verify the ‘only if’ direction in the general form of Farkas’ lemma. (1pt)

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$$\begin{aligned} 0 &> y_0 b_0 + y_1 b_1 \\ &= y_0(Px_0 + Ax_1) + y_1 b_1 \\ &\geqslant y_0(Px_0 + Ax_1) + y_1(Qx_0 + Bx_1) \end{aligned}$$

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there exists $x = (x_0, x_1)$ s.t.

$$Px_0 + Ax_1 = b_0 \qquad (17)$$

$$Qx_0 + Bx_1 \leqslant b_1 \qquad (18)$$

$$x_1 \geqslant 0 \qquad (19)$$

We show that at most one of $x = (x_0, x_1)$ and $y = (y_0, y_1)$ exists. Suppose by the contrary that both $x = (x_0, x_1)$ and $y = (y_0, y_1)$ exists. Then

$$\begin{aligned} 0 &> y_0 b_0 + y_1 b_1 \\ &= y_0(Px_0 + Ax_1) + y_1 b_1 \\ &\geqslant y_0(Px_0 + Ax_1) + y_1(Qx_0 + Bx_1) \\ &= (y_0 P + y_1 Q)x_0 + (y_0 A + y_1 B)x_1 \\ &= 0 + (y_0 A + y_1 B)x_1 \\ &\geqslant 0, \end{aligned}$$

which is a contradiction.

Question 5

Assume that both (P) and (D) has a solution in the duality theorem. Prove that weak duality holds, that is, max $\leqslant$ min. (1pt)

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$$\max c_0 x_0 + c_1 x_1 \tag{20}$$
$$\text{s.t. } P x_0 + A x_1 = b_0 \tag{21}$$
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Dual:

$$\min y_0 b_0 + y_1 b_1 \tag{24}$$
$$\text{s.t. } y_0 P + y_1 Q = c_0 \tag{25}$$
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Let $x = (x_0, x_1)$ be a solution of (P), and $y = (y_0, y_1)$ a solution of (D).

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$$c_0x_0 + c_1x_1 = (y_0P + y_1Q)x_0 + c_1x_1$$

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$$\begin{aligned} c_0 x_0 + c_1 x_1 &= (y_0 P + y_1 Q) x_0 + c_1 x_1 \\ &\leqslant (y_0 P + y_1 Q) x_0 + (y_0 A + y_1 B) x_1 \end{aligned}$$

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Question 6

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c_1, \ldots, c_k \in \mathbb{R}^n$ Formulate the following problem as an LP: $Ax = b, x \geqslant 0, \min f(x)$ where $f(x) := \max\{c_1 x, \ldots, c_k x\}$. (1pt)

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Question 7

Reduce the following systems of inequalities to each other (in the sense that if we can solve one of them, then we can solve any of them):

(I): $\begin{aligned} Ax &= b \\ x &\geqslant 0 \end{aligned}$ **(II):** $\begin{aligned} Bx &\leqslant b \\ x &\geqslant 0 \end{aligned}$ **(III):** $Qx \leqslant b$ **(IV):** $\begin{aligned} Px_0 &= b_0 \\ Qx_1 &\leqslant b_1 \end{aligned}$

Write up Farkas' lemma for all of them. (3pts)

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We know how to solve (III), and we would like to solve (IV).

Let us define $Q' := \begin{bmatrix} \text{P} & 0 \\ \text{-P} & 0 \\ 0 & \text{Q} \end{bmatrix}$, $b = (b_0, -b_0, b_1)$, and $x' = (x_0, x_1)$.

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Then $Q'x' \leqslant b'$, is equivalent to (IV).

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Then $Q'x' \leqslant b'$, is equivalent to (IV).

We know how to solve (IV), and we would like to solve (I).

Let us define $P' := A$, $Q' = -Id$, $b'_0 = b$, $b'_1 = 0$, and $x'_0 = x$, $x'_1 = x$.

(I): $Ax = b$
$x \geqslant 0$

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(IV): $Px_0 = b_0$
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We know how to solve (IV), and we would like to solve (I).

Let us define $P' := A$, $Q' = -Id$, $b'_0 = b$, $b'_1 = 0$, and $x'_0 = x$, $x'_1 = x$.

Then $P'x'_0 = b'_0$, $Q'x'_1 \leqslant b'_1$, is equivalent to (IV).

We write the Farkas' Lemma for each problem:

(I). There exists x s.t. $\begin{aligned} Ax &= b \\ x &\geqslant 0 \end{aligned}$ $\iff$ there is no y s.t. $\begin{aligned} yA &\geqslant 0 \\ yb &< 0 \end{aligned}$

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(II). There exists x s.t. $Bx \leqslant b$, $x \geqslant 0$ $\iff$ there is no y s.t. $yB \geqslant 0$, $y \geqslant 0$, $yb < 0$

$$\begin{aligned} Bx &\leqslant b \\ x &\geqslant 0 \end{aligned} \quad\iff\quad \begin{aligned} yB &\geqslant 0 \\ y &\geqslant 0 \\ yb &< 0 \end{aligned}$$

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(III). There exists x s.t.

$$Qx \leqslant b \quad\iff\quad \text{there is no } y \text{ s.t. } \begin{aligned} yQ &= 0 \\ y &\geqslant 0 \\ yb &< 0 \end{aligned}$$

(IV). There exists x s.t.

$$\begin{aligned} Px_0 &= b_0 \\ Qx_1 &\leqslant b_1 \end{aligned} \quad\iff\quad \text{there is no } y \text{ s.t. } \begin{aligned} y_0 P + y_1 Q &= 0 \\ y_1 &\geqslant 0 \\ y_0 b_0 + y_1 b_1 &< 0 \end{aligned}$$