Continuous Optimization
Date: 11 November 2021
Submission deadline:
18 November, 11:00

Exercise 1 (1pt)**.** Let $f(x) = x^2 - a$. Show that Newton's method leads to the recurrence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Exercise 2 (2pts)**.** Consider the problem of minimizing the function $f(x) = x \log x$ over $x \in \mathbb{R}_{>0}$. Perform the full convergence analysis of Newton's method - consider all starting points $x_0 \in \mathbb{R}_{>0}$ and determine where the method converges for each of them.

Exercise 3 (2pts)**.**

(a) Let $f(x_1, x_2) = x_1^2 + Kx_2^2$ be a quadratic function where $K > 0$. Define $\|(u_1, u_2)\|_\circ := \sqrt{u_1^2 + Ku_2^2}$. Determine the optimum point of $\max_{\|u\|_\circ = 1} -\langle \nabla f(x), u \rangle$.

(b) Let $f(x) = x^T Ax + b^T x$ (where A is positive definite). Define $\|u\|_A := \sqrt{u^T Au}$. Determine the optimum point of $\max_{\|u\|_A = 1} -\langle \nabla f(x), u \rangle$.

Exercise 4 (1pt)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ a function and define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ as $\tilde{f}(x) = f(Ax + b)$ where $A \in \mathbb{R}^{n \times n}$ is an invertible matrix and $b \in \mathbb{R}^n$. Verify that if x_0 moves to x_1 by applying one step of Newton's method with respect to $\tilde{f}$, then $y_0 = Ax_0 + b$ moves to $y_1 = Ax_1 + b$ by applying one step of Newton's method with respect to f.

Exercise 5 (1pt)**.** Verify that $\|n(x)\|_x$ is affinely invariant.

Exercise 6 (1pt)**.** Verify that the **NL** condition is affinely invariant, that is, $x \mapsto f(x)$ satisfies this condition if and only if $x \mapsto f(\Phi(x))$ satisfies it, for any affine change of variables Φ.

Exercise 7 (2pts)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Verify that $\|\cdot\|_{H(x)^{-1}}$ is dual to the norm $\|\cdot\|_x$ (H denotes the Hessian of f).

Exercise 8 (2pts)**.** Suppose $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $B \in \mathbb{R}^{n \times n}$ is symmetric such that $-\alpha A \preceq B \preceq \alpha A$ for some $\alpha > 0$. Prove that $\|A^{-1/2} B A^{-1/2}\| \leq \alpha$.