King-serf duo by monochromatic paths in k-edge-coloured tournaments

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Abstract

An open conjecture of Erdős states that for every positive integer k there is a (least) positive integer $f(k)$ so that whenever a tournament has its edges colored with k colors, there exists a set S of at most $f(k)$ vertices so that every vertex has a monochromatic path to some point in S. We consider a related question and show that for every (finite or infinite) cardinal $\kappa > 0$ there is a cardinal λ_κ such that in every κ-edge-coloured tournament there exist disjoint vertex sets K, S with total size at most λ_κ so that every vertex v has a monochromatic path of length at most two from K to v or from v to S.

1 Introduction

A **tournament** $T = (V(T), A(T))$ is a directed graph obtained by orienting the edge set of a (possibly infinite) complete undirected graph. A directed cycle is called a **dicycle** for short. We use some basic set theoretic conventions. We consider functions f as sets of ordered pairs where $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$ imply $y = z$. For a finite or infinite cardinal κ let $\exp_0(\kappa) = \kappa$ and let $\mathbf{exp_{k+1}(\kappa)} = 2^{\exp_k(\kappa)}$. Remember that a cardinal is the set of the ordinals that are smaller than itself, for example $3 = \{0, 1, 2\}$. A $\boldsymbol{\kappa}$**-edge-colouring** of a tournament T is a function $c : A(T) \to \kappa$. A **monochromatic path** is a directed path (repetition of vertices is not allowed) with edges having the same colour. We call a dicycle **quasi-monochromatic** if all but at most one of its edges have the same colour.

Our investigation was motivated by the following conjecture of Erdős [15, p. 274].

Conjecture 1.1 (Erdős). *For every positive integer k there is a (least) positive integer $f(k)$ so that every k-edge-coloured finite tournament admits a subset $S \subseteq V(T)$ of size at most $f(k)$ such that S is reachable from every vertex by a monochromatic path.*

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It is known that $f(1) = f(2) = 1$, and there is an example showing that $f(3) \geq 3$ (see [15]). However, there is no known constant upper bound for $f(3)$, although it is conjectured to be 3 by Erdős. As a weakening of the original conjecture, we consider source-sink pairs instead of one sink set S. However, we may add bounds on the length of the monochromatic paths. More precisely, a **king-serf duo by monochromatic paths** consists of disjoint vertex sets $K, S \subseteq V(D)$ so that every vertex v has a monochromatic path of length at most two from K to v or from v to S. The **size** of the duo is defined as $|K| + |S|$. An edge uv of an edge-coloured tournament T is called **forbidding** if there is no monochromatic path of length at most two from v to u. Note that if T' is a subtournament of T containing a forbidding edge uv, then uv is forbidding edge with respect to T' as well.

The main result of the paper is the following.

Theorem 1.2. *For every (finite or infinite) cardinal κ there is a cardinal $\lambda_\kappa \leq \mathsf{exp}_{10}(\kappa)$ such that in every κ-edge-coloured tournament there exists a king-serf duo by monochromatic paths of size at most λ_k. For finite κ one can guarantee $\lambda_\kappa \leq \kappa^{62500\kappa}$.*

The rest of the paper is organized as follows. In Section 2, we give an overview of previous results. Theorem 1.2 is then proved in Section 3.

2 Previous work

Given a digraph $D = (V, A)$, an independent set $K \subseteq V$ is called a kernel if it is absorbing, that is, there exists a directed edge from K to v for every $v \in V - K$. Kernels were introduced by Von Neumann and Morgenstern [14] in relation to game theory.

The concept of kernels was generalized by Galeana-Sánchez [5] for edge-coloured digraphs. In the coloured case, independence and absorbency are only required by means of monochromatic paths, hence these sets are called kernels by monochromatic paths. The existence of such kernels is widely studied, see [6, 7, 8, 9, 10, 11, 13]. The case when K is an absorbing set but not necessarily independent by monochromatic paths is also of interest. Since an absorbing set always exists in a k-coloured digraph, a natural problem is to find one with minimum size, which motivates the conjecture of Erdős (Conjecture 1.1). In [15], Sands, Sauer and Woodrow proved that every 2-edge-coloured tournament admits an absorbing vertex, and also presented a 3-edge-coloured tournament in which the minimum size of an absorbing set is 3. They conjectured that every 3-edge-coloured tournament without polychromatic dicycles of length 3 has an absorbing vertex. Minggang [13] verified a slightly different version of the conjecture claiming that any k-edge-coloured tournament without polychromatic -not necessarily directed- cycles of length 3 contains an absorbing vertex. Meanwhile, examples show that for every $k \geq 5$, there exists a k-edge-coloured tournament without polychromatic dicycle of length 3 without an absorbing vertex. Galeana-Sánchez [5] proved that if each directed cycle of length at most 4 in a k-edge-coloured tournament T is quasi-monochromatic then T has an absorbing vertex. In his PhD thesis [1], Bland provided several sufficient conditions for the existence of an absorbing vertex

in a k-edge-coloured tournament. He also gave a sufficient condition for the existence of an absorbing set of size 3 in 3-edge-coloured tournaments.

Quasi-kernels are possible weakenings of kernels. An independent set $K \subseteq V$ is a quasi-kernel if for each vertex $v \in V - K$ there exists a path of length at most 2 from K to v (quasi-sink sets can be defined analogously). The fundamental theorem of Chvátal and Lovász [2] shows that every finite digraph contains a quasi-kernel. In [3], P.L. Erdős and Soukup studied the existence of quasi-kernels in infinite digraphs. As the plain generalization of the Chvátal-Lovász theorem fails even for tournaments, they considered the problem of finding a partition $V = V_1 \cup V_2$ of the vertex set such that the induced subgraph $D[V_1]$ has a quasi-kernel and $D[V_2]$ has a quasi-sink. The authors conjectured that such a partition exists for any (possibly infinite) digraph. They verified that every (possibly infinite) directed graph $D = (V, A)$ contains two disjoint, independent subsets K and S of V such that for each node $v \in V$ there exists a path of length at most 2 from K to v or from v to S, but the conjecture is still open.

The motivation of our investigations was to combine the notions of absorbing sets by monochromatic paths and that of quasi-kernels and sinks, which lead to the definition of a king-serf duo by monochromatic paths, and to prove an analogue of Conjecture 1.1.

3 Proof of Theorem 1.2

The proof relies on the following theorem due to Erdős, Hajnal and Pósa [4] (finite case) and Hajnal [12] (infinite case).

Theorem 3.1 (Erdős, Hajnal and Pósa). *For every finite simple graph H and cardinal $\kappa > 0$ there is a simple graph G of size at most $\mathsf{exp}_{|V(H)|+5}$ (at most $\kappa^{500|V(H)|^3\kappa}$ in the finite case) such that in any κ-edge-colouring of G one can find a monochromatic induced subgraph isomorphic to H.*

With the help of Theorem 3.1, first we prove the following.

Lemma 3.2. *For every cardinal $\kappa > 0$ there exists a tournament T_κ of size at most $\mathsf{exp}_{10}(\kappa)$ (at most $\kappa^{62500\kappa}$ in the finite case) such that in any κ-edge-colouring of T_κ there exists a quasi-monochromatic dicycle of length three.*

Proof. Pick a graph G ensured by Theorem 3.1 for κ and $H = C_5$, that is, a cycle of length 5. Fix a well-ordering of $V(G)$. Let T_κ denote the tournament obtained by orienting the edges of G forward according to the ordering, and by adding all missing edges as backward edges. We claim that T_κ satisfies the conditions of the lemma.

Take an arbitrary κ-edge-colouring of T_κ. The choice of G implies that there is a monochromatic (not necessarily directed) cycle C of length 5 in the graph such that $A(C)$ consists of forward edges, and all the other edges induced by $V(C)$ in T_κ are backward edges.

No matter how the edges of C are oriented, we can always find a directed path of length two in $A(C)$. Take such a path, say uv and vw. These edges together with wu form a quasi-monochromatic dicycle, concluding the proof of the lemma. $\qquad\square$

We claim that $\lambda_\kappa := |V(T_\kappa)|$ satisfies the conditions of the theorem. Suppose to the contrary that there exists a κ-edge-coloured tournament T not containing a king-serf duo by monochromatic paths of size at most λ_κ. Let T_κ be a tournament that we obtain by applying Lemma 3.2.

Lemma 3.3. *T has a subtournament isomorphic to T_κ consisting of forbidding edges.*

Proof. We build up the desired subtournament by transfinite recursion. Let $V(T_\kappa) = \{u_\gamma\}_{\gamma < |V(T_\kappa)|}$. Assume that for some $\alpha < |V(T_\kappa)|$ we have already found an $\subset$-increasing chain $\langle f_\beta : \beta < \alpha \rangle$ of $T_\kappa \to T$ embeddings where $\mathsf{dom}(f_\beta) = \{u_\gamma\}_{\gamma < \beta}$ and the images of the edges of T_κ are forbidding edges of T. If β is a limit ordinal, we may simply take $f_\beta := \bigcup_{\gamma < \beta} f_\gamma$ to keep the conditions. Assume that $\beta = \delta + 1$. Let $O = \{\gamma < \delta : u_\delta u_\gamma \in A(T_\kappa)\}$. As T is a counterexample, the sets $K := \{f_\delta(u_\gamma)\}_{\gamma \in O}$ and $S := \{f_\delta(u_\gamma)\}_{\gamma \in \delta \setminus O}$ cannot form a king-serf duo by monochromatic paths. Therefore there is a vertex $v \in V(T)$ such that there is a forbidding edge from v to every element of K, and there is a forbidding edge from every element of S to v. But then $f_{\delta+1} := f_\delta \cup \{\langle u_\delta, v \rangle\}$ maintains the conditions. Finally, the image of $f := \bigcup_{\gamma < |V(T_\kappa)|}$ gives the desired copy of T_κ. $\square$

The κ-edge-colouring of T defines a κ-edge-colouring of its T_κ subgraph as well. Therefore, by the choice of T_κ, there is a quasi-monochromatic dicycle C of length three in T_κ. Let uv denote the edge of C with different colour than the others if C contains two colours, and let uv be an arbitrary edge of C if it is monochromatic. Then $C - uv$ is a monochromatic path of length two from v to u, contradicting uv being a forbidding edge of T. This finishes the proof of Theorem 1.2.

Acknowledgements

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